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Jörg Kortemeyer

Complex Numbers

An Introduction for First Year Students

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Complex Numbers

An Introduction for First Year Students

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What You Can Find in This *essential*

- Basics of complex numbers: Introduction of all three representations (Cartesian representation, polar form, Euler form) as well as the geometric notion.
- Methods for conversion between the three representations.
- Numerous exemplary tasks on complex numbers.
- Method for calculating the n zeros of a polynomial of the n-th degree.

Dedicated to my students at the Leibniz Universität Hanover (Leibniz University of Hanover), the Universität Paderborn (Paderborn University) as well as the Technische Universität Clausthal (Clausthal University of Technology), who have always contributed to the improvement of my teaching by their questions.

Preface

This *essential* provides an introduction to complex numbers. First, it explains what a complex number is and motivates the introduction of this number range extension. Then, the four basic arithmetic operations, as well as the exponentiation of complex numbers, are explained in more detail, whereby the usual three representations (Cartesian representation, polar form, Euler form) are introduced. These are explained with the help of many examples. The last section deals with Moivre's theorem for taking roots of complex numbers.

The *essential* is aimed at:

- Students of engineering sciences as well as students of natural sciences for a better understanding of the topic.
- STEM study beginners to repeat and expand their school knowledge with a view to a successful entry into studies
- Pupils who want to study more advanced topics which have high relevance for many courses of study.

I wish the readers a lot of fun with the topic "complex numbers".

Technische Universität Clausthal
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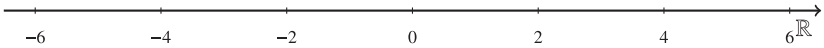
Why Complex Numbers?

1

? What numbers do we know so far and what were the reasons for introducing further number ranges?

Over the years, new number ranges are introduced in school again and again. This begins with the introduction of natural numbers $\mathbb{N}$ as the set of numbers in counting, i.e. $\{1, 2, 3, 4 \ldots\}$, where it quickly becomes apparent that one can already leave this set when subtracting two natural numbers. For example, the solution of $3 - 5$ is not a natural number, although 3 and 5 are natural numbers. Hence, the integers $\mathbb{Z}$ are introduced which now also contain the number 0 and negative numbers, i.e. the set $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$.

By division of two integers again this set can be left, so that $\mathbb{Q}$, the set of fractions or "the rational numbers", must be introduced, in order to be able to represent all results of divisions. For example, taking square roots of non-negative numbers can again produce numbers that cannot be represented as fractions and are therefore not part of the rational numbers $\mathbb{Q}$. For this reason, the set of real numbers $\mathbb{R}$ is introduced. Every number can be represented on the so-called number line (see the following figure) and this line can be covered without gaps by real numbers:



Which tasks are still not solvable? This starts with a relatively simple equation, namely $x^2 + 1 = 0$, which has no real solutions. Our aim now is to find solutions for such equations too, which will lie in the set of complex numbers. For this purpose, we first make further considerations.

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What belongs to the real numbers?

We have a rough idea of all the numbers that can be represented as fractions, i.e. the set of rational numbers $\mathbb{Q}$. Which numbers of the set of real numbers, i.e. $\mathbb{R}$, are irrational?

In the set of irrational numbers, we distinguish two types of numbers, namely algebraic and transcendental numbers: Algebraic numbers are zeros of polynomials with integer coefficients. Examples of these are $\sqrt{2}$ as a solution of $x^2 - 2 = 0$, $\sqrt{3}$ as a solution of $x^2 - 3 = 0$, or also $\sqrt[17]{42}$ as a solution of $x^{17} - 42 = 0$. The much larger group is the transcendental numbers, which are not solutions of integer polynomials. These include the numbers π or e, which, however, can be approximated, for example, by fractions. Thus, applies: $\pi \approx \frac{22}{7}$.

? Where are the complex numbers if the real numbers already cover the entire number line?

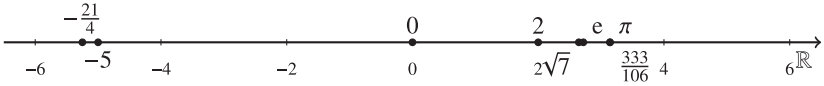
As with natural numbers, integers and rational numbers, we can specify so-called relations for real numbers, i.e. whether the following applies to two real numbers a and b: $a < b$, $a = b$ or $a > b$. This is called an **order property**. By way of example, let us consider a set of real numbers:

$$2, 0, -5, \frac{333}{106}, -\frac{21}{4}, \sqrt{7}, \pi \text{ and } \mathrm{e} \tag{1.1}$$

The following applies to the numbers given

$$-\frac{21}{4} < -5 < 0 < 2 < \sqrt{7} < \mathrm{e} < \frac{333}{106} < \pi \tag{1.2}$$

The number line has a direction, which is indicated by an arrow. In the following, we will consider it as the *x-axis* and there 0 as the so-called **origin**. Then, on the left, are the negative numbers < 0 and on the right, the positive numbers > 0. In this way of looking at it, we can draw in all eight numbers:



But how can we now classify, for example, the two solutions from $x^2 + 1 = 0$ that are not real? If we multiply a positive number by itself, the result is a positive

number. However, the same is true for a negative number, and furthermore $0^2 = 0$. So, for all real numbers x, it is true that $x^2 \geq 0$, so the square of a real number is non-negative. In this Essential, we deal with numbers whose square can be negative. Thus, we come to the numbers $\pm i$ whose square is -1 in each case. These numbers are sometimes called "imaginary", but they actually occur in applications, for example, in "complex alternating current calculus" in electrical engineering. Our aim in Chap. 2 is first the transfer and extension of known rules from the real numbers. The traceability to real numbers simplifies many considerations and this Essential introduces in more detail how such transfers can be chosen.

About the zeros of polynomials

A **polynomial** is a function $f(x)$ of the form

$$f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_1 x + a_0 \text{ with } x, a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{R} \tag{1.3}$$

Here, n is called the **degree of** the polynomial. Thus, the degree of $f(x) = x^2 + 1$ is equal to 2. If one examines the zeros in the real numbers, one can make different statements for even n and for odd n:

- n even: The polynomial $f(x)$ always has an even number of zeros and at most n. In particular, it cannot have any zeros like our example $x^2 + 1$.
- n odd: The polynomial $f(x)$ always has an odd number of zeros and at most n. This also means that it must have at least one real zero. This can be seen by plotting $f(x)$ and examining it more closely because if $f(x)$ takes on positive function values for very large x, it must take on negative function values for very small x. Similarly, if you have negative function values for very large x, then the function values for very small x must be positive. Thus, $f(x)$ must cross the *x-axis* somewhere in between, that is, it must have a zero. Related keywords: Continuity of polynomials, intermediate value theorem

It would be desirable to delete the "at most" in the last two statements, i.e. to always have exactly as many zeros as the degree of the polynomial. Using complex numbers, this is indeed the case. How to find these zeros is explained by Moivre's theorem presented in Chap. 4.

But what do decompositions of higher degree polynomials look like? For this we will consider an example of a polynomial of degree 12, which will accompany us through this Essential:

? What are the solutions of $x^{12} - 1 = 0$?

It holds with the third binomial formula:

$$x^{12} - 1 = \left(x^6 - 1\right)\left(x^6 + 1\right) \tag{1.4}$$

Now, we can see that $x^6 - 1$ has ± 1 as zeros and is therefore divisible by $x^2 - 1$. Analogously, the $x^6 + 1$ has $\pm \mathrm{i}$ as zeros and is therefore divisible by $x^2 + 1$. We obtain via polynomial division:

$$x^6 - 1 = \left(x^2 - 1\right)\left(x^4 + x^2 + 1\right) \tag{1.5}$$

and

$$x^6 + 1 = \left(x^2 + 1\right)\left(x^4 - x^2 + 1\right) \tag{1.6}$$

with suitable completions of the squares and third binomial formula, one can now further decompose the two fourth-degree polynomials:

$$x^4 + x^2 + 1 = \left(x^2 + 1\right)^2 - x^2 = \left(x^2 + 1 - x\right)\left(x^2 + 1 + x\right) \tag{1.7}$$

$$x^4 - x^2 + 1 = \left(x^2 + 1\right)^2 - 3x^2 = \left(x^2 + 1 - \sqrt{3}x\right)\left(x^2 + 1 + \sqrt{3}x\right) \tag{1.8}$$

In total, we have now decomposed the twelfth-degree polynomial into six-second degree polynomials:

$$\left(x^2 - 1\right)\left(x^2 + 1 - x\right)\left(x^2 + 1 + x\right)\left(x^2 + 1\right)\left(x^2 + 1 - \sqrt{3}x\right)\left(x^2 + 1 + \sqrt{3}x\right) \tag{1.9}$$

We realize that the zeros of $x^2 - 1$ are the real numbers ± 1. However, for the five remaining quadratic equations, there are no real solutions or, when applying the pq- formula, each time the expression under the root becomes negative. We will revisit this task in Sect. 4.1.

Check for updates

2 Cartesian Representation—Algebra and Geometry of Complex Numbers

2.1 Introduction of the Representation Form

Definition 2.1
A **complex** number is an ordered pair of real numbers, which is usually referred to as z or w. For $a, b \in \mathbb{R}$, we can describe a complex number as:

$$z = a + \mathrm{i} \cdot b \tag{2.1}$$

i denotes a number for which the rule applies $\mathrm{i}^2 = -1$. We denote the set of complex numbers by $\mathbb{C}$.

! $\mathrm{i} \neq \sqrt{-1}$

Assuming $\mathrm{i} = \sqrt{-1}$, we get the following equation:

$$1 = \sqrt{1} = \sqrt{1 \cdot 1} = \sqrt{(-1) \cdot (-1)} = \sqrt{-1} \cdot \sqrt{-1} = \mathrm{i} \cdot \mathrm{i} = \mathrm{i}^2 = -1 \tag{2.2}$$

We realize that under this assumption one can show the contradiction $1 = -1$. For this reason, one sets i as a solution of $z^2 = -1$, so holds $z = \pm\mathrm{i}$.

We will look at the root of negative numbers in more detail later. With the help of the introduced notation, we can write:

$$\sqrt{-9} = \sqrt{9 \cdot (-1)} = \pm 3\mathrm{i}. \tag{2.3}$$

? **What are the powers of i?**

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We already know that $\mathrm{i}^2 = -1$ holds. Then, we get according to power laws:

$$\mathrm{i}^3 = \mathrm{i}^2 \cdot \mathrm{i} = (-1) \cdot \mathrm{i} = -\mathrm{i} \tag{2.4}$$

For the calculation of i^4, we can approach in two ways. For example, it is

$$\mathrm{i}^4 = \mathrm{i}^2 \cdot \mathrm{i}^2 = (-1) \cdot (-1) = 1. \tag{2.5}$$

Alternatively, we can also calculate:

$$\mathrm{i}^4 = \mathrm{i}^3 \cdot \mathrm{i} = (-\mathrm{i}) \cdot \mathrm{i} = (-1) \cdot \mathrm{i}^2 = 1. \tag{2.6}$$

So, we have found that $\mathrm{i}^4 = 1$. Thus, using power laws, we can determine i^5, i^6 and i^7. We obtain:

- $\mathrm{i}^5 = \mathrm{i}^{1+4} = \mathrm{i}^1 \cdot \mathrm{i}^4 = \mathrm{i}^1 = \mathrm{i}$
- $\mathrm{i}^6 = \mathrm{i}^{2+4} = \mathrm{i}^2 \cdot \mathrm{i}^4 = \mathrm{i}^2 = -1$
- $\mathrm{i}^7 = \mathrm{i}^{3+4} = \mathrm{i}^3 \cdot \mathrm{i}^4 = \mathrm{i}^3 = -\mathrm{i}$

We now see a pattern: By splitting off the summand 4 in the exponent, we can translate higher powers of i to lower powers. This is also true for higher powers $m = r + 4n$. Thus, we can write:

$$\mathrm{i}^m = \mathrm{i}^{r+4n} = \mathrm{i}^r \cdot \mathrm{i}^{4n} = \mathrm{i}^r \cdot (\mathrm{i}^4)^n = \mathrm{i}^r \cdot 1^n = \mathrm{i}^r \tag{2.7}$$

This can also be continued into the negative range. For example:

$$\mathrm{i}^{-1} = \frac{1}{\mathrm{i}} = \frac{\mathrm{i}^4}{\mathrm{i}} = \mathrm{i}^3 = -\mathrm{i} \tag{2.8}$$

> Powers of i

For the number i, the following equation holds:

$$\mathrm{i}^2 = -1. \tag{2.9}$$

Using power laws, the result is an alternation of four values when exponentiating i:

$$\mathrm{i}, \mathrm{i}^2 = -1, \mathrm{i}^3 = -\mathrm{i}, \mathrm{i}^4 = 1, \mathrm{i}^5 = \mathrm{i}, \mathrm{i}^6 = -1, \mathrm{i}^7 = -\mathrm{i}, \mathrm{i}^8 = 1, \mathrm{i}, -1, -\mathrm{i}, 1, \mathrm{i}, -1, -\mathrm{i}, 1, \ldots \tag{2.10}$$

For $z = a + \mathrm{i}b$, number a is the **real part** or $\mathrm{Re}(z)$ and b is the **imaginary part** of z or $\mathrm{Im}(z)$. The numbers with $\mathrm{Im}(z) = 0$ are exactly the real numbers.

> Real part and imaginary part

For $z = a + \mathrm{i}b$, $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$. Both the real part and the imaginary part are **real** numbers.

Definition 2.2

(a) Two complex numbers $z = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$ are **equal** if both their real parts and their imaginary parts are equal. Thus, it holds:

$$a = c \text{ and } b = d \tag{2.11}$$

(b) The **absolute value** of a complex number $z = a + \mathrm{i}b$, written as |z|, is defined as

$$|z| = \sqrt{a^2 + b^2} \tag{2.12}$$

The absolute value of a complex number is always a non-negative real number.

? Calculate $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ and $|z|$ for $z = 12 - 5\mathrm{i}$

We can read off the real and imaginary parts directly: $\mathrm{Re}(z) = 12$ and $\mathrm{Im}(z) = -5$. For |z|, it follows according to Eq. (2.12):

$$|z| = \sqrt{12^2 + (-5)^2} = \sqrt{144 + 25} = \sqrt{169} = 13 \tag{2.13}$$

Definition 2.3

For a complex number $z = a + \mathrm{i}b$, the **complex conjugate** of z, denotation $\overline{z}$, is defined as $\overline{z} = a - \mathrm{i}b$. For example, for $z = 4 - \mathrm{i}$, it holds $\overline{z} = 4 + \mathrm{i}$. There are also two special cases:

- for a purely real number z, i.e. $\mathrm{Im}(z) = 0$, holds: $\overline{z} = z$
- for a purely imaginary number z, i.e. $\mathrm{Re}(z) = 0$, holds: $\overline{z} = -z$, i.e., for example, for $z = 3\mathrm{i}$, $\overline{z} = -3\mathrm{i}$.

2.2 Calculations with Complex Numbers

This section presents typical tasks and the underlying theory that arise when computing with complex numbers in Cartesian representation.

2.2.1 Basic Arithmetic Operations

The basic arithmetic operations, i.e. addition, subtraction, multiplication and division, can be performed on complex numbers in a very similar way to real numbers if we consider i as a variable whose square is -1.

Theorem 2.1
Let z and w be any two complex numbers, that is, $z = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$. Then, it holds:

(a) $z + w = (a + c) + \mathrm{i}(b + d)$
(b) $z - w = (a - c) + \mathrm{i}(b - d)$
(c) $z \cdot w = ac - bd + \mathrm{i}(ad + bc)$

Proof
(a) and (b) follow directly from calculating with polynomials. For (c):

$$z \cdot w = (a + \mathrm{i}b) \cdot (c + \mathrm{i}d) = ac + a\mathrm{i}d + \mathrm{i}bc + \mathrm{i}^2 bd = ac + a\mathrm{i}d + \mathrm{i}bc - bd = ac - bd + \mathrm{i}(ad + bc)$$

? Calculate for $z = 1 - 2\mathrm{i}$ and $w = 5 + \frac{1}{2}\mathrm{i}$ the numbers $z + 4w$, $|z - 2w|$, $z \cdot w$ und w^2

- $z + 4w = 1 - 2\mathrm{i} + 4\left(5 + \frac{1}{2}\mathrm{i}\right) = 21$
- $|z - 2w| = \sqrt{(1 - 10)^2 + (-2 - 1)^2} = \sqrt{9^2 + 9} = \sqrt{90} = 3\sqrt{10}$
- $z \cdot w = (1 - 2\mathrm{i}) \cdot \left(5 + \frac{1}{2}\mathrm{i}\right) = 5 + \frac{1}{2}\mathrm{i} - 10\mathrm{i} + 1 = 6 - \frac{19}{2}\mathrm{i}$
- $w^2 = \left(5 + \frac{1}{2}\mathrm{i}\right) \cdot \left(5 + \frac{1}{2}\mathrm{i}\right) = 25 + 2 \cdot 5 \cdot \frac{1}{2}\mathrm{i} - \frac{1}{4} = \frac{99}{4} + 5\mathrm{i}$

! The square of a complex number is not necessarily a non-negative number

This rule, which holds for real numbers and can be used to check results, no longer holds in the complex numbers. For example, for $z = -2 - 5\mathrm{i}$:

$$(-2 + 5\mathrm{i})^2 = (-2 + 5\mathrm{i})(-2 + 5\mathrm{i}) = 4 - 10\mathrm{i} - 10\mathrm{i} - 25 = -21 - 20\mathrm{i} \quad (2.14)$$

$z \cdot \overline{z}$ is always a real number:

$$\begin{aligned} z \cdot \overline{z} = (a + \mathrm{i}b)(a - \mathrm{i}b) &= a^2 - \mathrm{i}ab + \mathrm{i}ab - \mathrm{i}^2 b \\ &= a^2 - \mathrm{i}^2 b^2 = a^2 + b^2 \end{aligned}$$

We will now use this result to divide complex numbers:

Theorem 2.2
(Division of complex numbers) Let $z = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$ be two complex numbers whose quotient is calculated, that is $\frac{z}{w} = \frac{a+\mathrm{i}b}{c+\mathrm{i}d}$. When dividing complex numbers, the fraction is expanded with the conjugate complex of the denominator:

$$\frac{z}{w} = \frac{a + \mathrm{i}b}{c + \mathrm{i}d} = \frac{a + \mathrm{i}b}{c + \mathrm{i}d} \cdot \frac{c - \mathrm{i}d}{c - \mathrm{i}d} = \frac{(ac + bd) + \mathrm{i}(bc - ad)}{c^2 + d^2} \tag{2.15}$$

! Typical error

You often see the following mistake in exams:

$$\frac{z}{w} = \frac{a + \mathrm{i}b}{c + \mathrm{i}d} = \frac{a}{c} + \mathrm{i}\frac{b}{d} \tag{2.16}$$

This transformation also does not correspond to the usual rules of fractional arithmetic.

? Calculate $z \cdot \overline{z}$ and $\frac{z}{w}$ for $z = -3 + 4\mathrm{i}$ and $w = 6 - 8\mathrm{i}$

We get

$$z \cdot \overline{z} = (-3 + 4\mathrm{i})(-3 - 4\mathrm{i}) = 9 + 16 = 25 \tag{2.17}$$

and

$$\frac{z}{w} = \frac{-3 + 4\mathrm{i}}{6 - 8\mathrm{i}} = \frac{(-3 + 4\mathrm{i})(6 + 8\mathrm{i})}{(6 - 8\mathrm{i})(6 + 8\mathrm{i})} = \frac{-18 - 24\mathrm{i} + 24\mathrm{i} - 32}{6^2 + 8^2} = \frac{-50}{100} = -\frac{1}{2} \tag{2.18}$$

The latter also follows directly from:

$$\frac{-3+4\mathrm{i}}{6-8\mathrm{i}} = \frac{-3+4\mathrm{i}}{(-2)(-3+4\mathrm{i})} = -\frac{1}{2} \tag{2.19}$$

2.2.2 Solutions of Polynomial Equations of Higher Degree

Polynomial equations can have real solutions, complex solutions, or a combination of real and complex solutions. The simplest case is quadratic equations like $x^2 + a^2 = 0$, which can be rearranged to $x^2 = -a^2$. Then, the solutions are

$$x_{1,2} = \pm\sqrt{-a^2} = \pm a \cdot \mathrm{i} \tag{2.20}$$

This can also be applied to more general equations $x^2 + px + q = 0$, whose solutions can be determined with the pq-formula $x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$.

? What are the solutions of $x^2 + 2x + 10 = 0$?

By means of the pq-formula, using Eq. (2.3), the following is obtained

$$x_{1,2} = -\frac{2}{2} \pm \sqrt{\left(\frac{2}{2}\right) - 10} = -1 \pm \sqrt{-9} = -1 \pm 3\mathrm{i} \tag{2.21}$$

The example shows that the two solutions are complex-conjugate to each other. Such pairs of complex-conjugate solutions always occur when all coefficients of the polynomial are real numbers and the number of real zeros is smaller than the degree of the polynomial.

> Solutions of polynomials with real coefficients

Solutions of polynomial equations with real coefficients are either real numbers or complex-conjugate pairs of numbers, that is, if a solution is $a + \mathrm{i}b$, then $a - \mathrm{i}b$ is too.

? Determine all solutions of $z^3 - 3z^2 + 4z - 2 = 0$. Tip: $z = 1 + \mathrm{i}$ is a zero.

One can start here with Horner's method and gets:

	1	-3	4	-2
$1+\mathrm{i}$		$1+\mathrm{i}$	$-3-\mathrm{i}$	2
	1	$-2+\mathrm{i}$	$1-\mathrm{i}$	0

So, the quotient is $z^2 + (-2+\mathrm{i}) \cdot z + (1-\mathrm{i})$. One can see that $z = 1$ is a zero, from which one can then determine the last zero:

	1	$-2+\mathrm{i}$	$1-\mathrm{i}$
1		1	$-1+\mathrm{i}$
	1	$-1+\mathrm{i}$	0

What we are looking for now is the zero of $z + (-1+\mathrm{i}) = 0$, i.e. $z = 1-\mathrm{i}$. Thus, all three zeros are determined.

Using the remark about solving polynomials with real coefficients, it is clear that with $z = 1+\mathrm{i}$, also $z = 1-\mathrm{i}$ is a zero. As a third-degree polynomial, the third zero must be real and since all the coefficients are integers, it must be a divisor of -2, i.e. $\pm 1, \pm 2$. By substituting, you can find $x = 1$ as the third zero.

2.2.3 Exponentiation of Complex Numbers in Cartesian Representation

In this section, a complex form of the binomial theorem is first introduced, followed by two examples, and then possible simplifications are presented.

Theorem 2.3
(Binomial theorem for complex numbers in Cartesian representation) Given $x, y \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then:

$$(x+\mathrm{i}y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (\mathrm{i}y)^k = \sum_{k=0}^{n} \binom{n}{k} x^k (\mathrm{i}y)^{n-k} \tag{2.22}$$

? What is $(1+\mathrm{i})^4$ and what is $(1+\mathrm{i})^6$?

(a) We can apply the binomial theorem from (2.22) with $a = b = 1$:

$$(1+\mathrm{i})^4 = 1 \cdot 1^4 \cdot \mathrm{i}^0 + 4 \cdot 1^3 \cdot \mathrm{i}^1 + 6 \cdot 1^2 \cdot \mathrm{i}^2 + 4 \cdot 1^1 \cdot \mathrm{i}^3 + 1 \cdot 1^0 \cdot \mathrm{i}^4 = 1 + 4\mathrm{i} - 6 - 4\mathrm{i} + 1 = -4$$

(b) Analogous to (a) it holds again $a = b = 1$. It follows (for a shorter notation, all powers of 1 are written directly as 1):

$$\begin{aligned}(1+\mathrm{i})^6 =& 1\cdot 1\cdot \mathrm{i}^0 + 6\cdot 1\cdot \mathrm{i}^1 + 15\cdot 1\cdot \mathrm{i}^2 + 20\cdot 1\cdot \mathrm{i}^3 + 15\cdot 1\cdot \mathrm{i}^4 + 6\cdot 1\cdot \mathrm{i}^5 + 1\cdot 1\cdot \mathrm{i}^6 \\ =& 1 + 6\mathrm{i} - 15 - 20\mathrm{i} + 15 + 6\mathrm{i} - 1 = -8\mathrm{i}\end{aligned}$$

It shows in each case that some terms cancel out others directly. As a simplification, an application of the power laws, in particular $a^{n\cdot m} = (a^n)^m$, is conceivable. So, we can transform (a) to $\left((1+\mathrm{i})^2\right)^2$ and get

$$\left((1+\mathrm{i})^2\right)^2 = (2\mathrm{i})^2 = -4 \tag{2.23}$$

(b) can then be interpreted on the one hand as

$$\left((1+\mathrm{i})^2\right)^3 = (2\mathrm{i})^3 = -8\mathrm{i} \tag{2.24}$$

or as a product

$$(1+\mathrm{i})^6 = (1+\mathrm{i})^4 \cdot (1+\mathrm{i})^2 = (-4)\cdot(2\mathrm{i}) = -8\mathrm{i} \tag{2.25}$$

> Quick exponentiation

Power laws can often be used instead of the binomial theorem to save effort. Because of $(\mathrm{i}-1)^2 = -1 - 2\mathrm{i} + 1 = -2\mathrm{i}$,

? What is $(\mathrm{i}-1)^9$, $\left(\frac{1+\mathrm{i}}{-\mathrm{i}}\right)^{12}$ and $(\sqrt{3}\mathrm{i}-1)^{10}$?

$$(\mathrm{i}-1)^8 = \left((\mathrm{i}-1)^2\right)^4 = (-2\mathrm{i})^4 = 16 \tag{2.26}$$

and thus

$$(\mathrm{i}-1)^9 = (\mathrm{i}-1)^8 \cdot (\mathrm{i}-1) = 16\mathrm{i} - 16 \tag{2.27}$$

it holds $\left(\frac{1+\mathrm{i}}{-\mathrm{i}}\right)^{12} = \left(\frac{(1+\mathrm{i})\cdot\mathrm{i}}{-\mathrm{i}\cdot\mathrm{i}}\right)^{12} = (\mathrm{i}-1)^{12}$. With the first part of the task follows:

$$(\mathrm{i}-1)^{12} = \left((\mathrm{i}-1)^2\right)^6 = (-2\mathrm{i})^6 = -64 \tag{2.28}$$

we calculate that

$$(\sqrt{3}\mathrm{i} - 1)^2 = -3 - 2\sqrt{3}\mathrm{i} + 1 = -2(1 + \sqrt{3}\mathrm{i}) \tag{2.29}$$

and

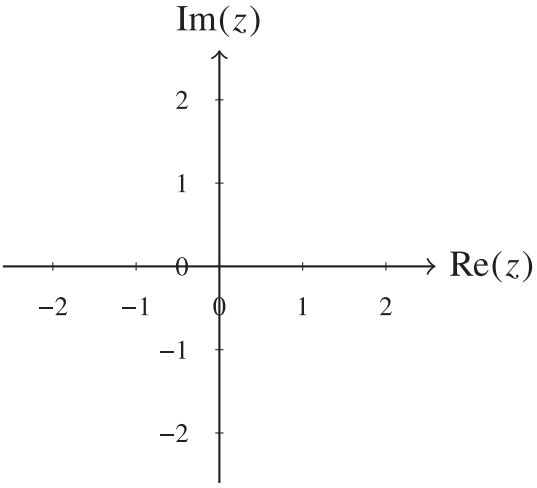
$$(\sqrt{3}\mathrm{i} - 1)^3 = (\sqrt{3}\mathrm{i} - 1)^2(\sqrt{3}\mathrm{i} - 1) = -2(1 + \sqrt{3}\mathrm{i})(\sqrt{3}\mathrm{i} - 1) = -2(-3 - 1) = 8. \tag{2.30}$$

This gives us:

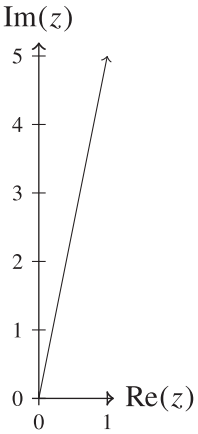
$$\begin{aligned}(\sqrt{3}\mathrm{i} - 1)^{10} &= \left((\sqrt{3}\mathrm{i} - 1)^3\right)^3 \cdot (\sqrt{3}\mathrm{i} - 1) \\ &= 8^3 \cdot (\sqrt{3}\mathrm{i} - 1) = 512 \cdot (\sqrt{3}\mathrm{i} - 1) = 512\sqrt{3}\mathrm{i} - 512\end{aligned}$$

2.3 Geometric Representations of Complex Numbers in the Gaussian Number Plane

In this section, we learn a useful way of representing complex numbers: Since a complex number can be written using two real numbers x and y, it can be directly transferred to a plane using a vector. As usual, the horizontal axis is the *x-axis* and the vertical axis is the *y-axis*. The *x-value* is the real part and the *y-value* the imaginary part:



In this form of representation, the complex number $z = 1 + 5\mathrm{i}$ is represented by a straight line that starts at the origin and ends at the point (1, 5). Analogously, $w = -2 + \mathrm{i}$ would be represented by a straight line from the origin to the point $(-2, 1)$. In such a representation, the plane is called the **Gaussian number plane**. The magnitude of z or w is here the length of the vector, which represents z or w.



In the following we transfer the rules for complex numbers, which we learned in the previous section: We know that for $z = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$ holds:

$$z + w = (a + c) + \mathrm{i}(b + d) \tag{2.31}$$

So, we add the two real parts and the two imaginary parts. Analogously, this happens in the same way with the addition and subtraction of two vectors, which is commutative. We get

$$\mathbf{v} + \mathbf{u} = (a + c) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (b + d) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{u} + \mathbf{v} \tag{2.32}$$

We realize that the addition or subtraction of complex numbers in Cartesian representation is analogous to the addition or subtraction of two-dimensional vectors. For multiplication and division, however, this representation seems less useful. For this reason, we introduce two more useful forms in the next chapter.

> **The complex numbers have no order property**

On the Gaussian number plane as a two-dimensional form of representation instead of a number line we recognize that we cannot specify relations, i.e. $<, =$ or $>$. Thus, we lose the order property, that is present in all other number domains introduced in Chapter 1, that is, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$. So, we cannot specify for $z, w \in \mathbb{C}$ whether it is $z < w$, $z = w$ or $z > w$. A remedy can be provided by considering their sums since these are real numbers that have the order property introduced in Chapter 1.

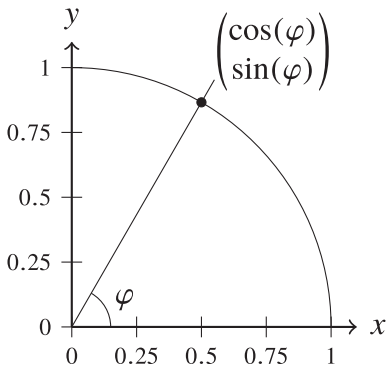
3 Two Further Representations: From the Polar Form to the Euler Form

3.1 Refresher Course on Trigonometry

Sine and cosine are usually introduced as aspect ratios in rectangular triangles. For the application in connection with complex numbers, however, the transfer to the unit circle is decisive.

> Unit circle and its relation to sine and cosine

The **unit circle** is the circle with radius 1 and the origin, that is, the point (0, 0), as the center. We now consider a straight line that has an angle of φ to the positive *x-axis:*



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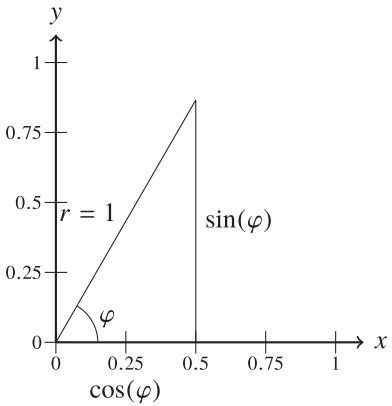
J. Kortemeyer, *Complex Numbers*, essentials,
https://doi.org/10.1007/978-3-658-34929-5_3

The intersection of this line with the unit circle is given by $\begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}$. Thus, a the decomposition yields $\cos(\varphi)$ as the horizontal part and $\sin(\varphi)$, as the vertical part of a connecting straight line of length 1 between the origin and the circular arc.

Connection to rectangular triangles



The sine is defined as the quotient of the length of the opposite cathetus and the length of the hypotenuse in a rectangular triangle. The cosine, in turn, is the quotient of the length of the ancathetus and the length of the hypotenuse in a rectangular triangle. Here, the hypotenuse length, which in this case is equal to the distance from the center of the circle to the edge of the circle, is equal to 1, since this length is equal to the radius. Thus, in each case, in the quotients for sine and cosine, the denominators become 1. Hence, for the sine, we get the length of the opposite cathetus, which is the distance from the *x-axis* to the *y-value* (that is, the *y-portion*). Similarly, for the cosine, we get the length of the adjacent side, which is the distance from the *y-axis* to the *x-value,* i.e. the *x-portion*.

But we can see something more by applying the Pythagorean theorem: It is true for any angle φ:

$$\sin(\varphi)^2 + \cos(\varphi)^2 = 1. \tag{3.1}$$

This statement is called **trigonometric Pythagoras.**

Definition 3.1
At universities, angles are usually given in radians. From the degree measure, you can convert as follows:

$$x^\circ \text{ corresponds to } y = \frac{x}{180} \cdot \pi \text{ in radian measure} \tag{3.2}$$

Thus, 180° in the degree measure is, for example, π in radians.

Periodicity and symmetry properties of sine and cosine

The sine and the cosine are 2π-periodic functions, i.e. the following applies

$$\sin(\varphi) = \sin(\varphi + 2k\pi), \qquad \cos(\varphi) = \cos(\varphi + 2k\pi) \qquad k \in \mathbb{Z} \tag{3.3}$$

The sine is point symmetric to the origin, i.e. for all angles φ applies:

$$\sin(-\varphi) = -\sin(\varphi) \tag{3.4}$$

The cosine is axisymmetric to the *y-axis*, i.e. applies to all angles φ:

$$\cos(\varphi) = \cos(-\varphi) \tag{3.5}$$

Angle table

Angle α	$0 \quad [= 0^\circ]$	$\frac{\pi}{6} \quad [= 30^\circ]$	$\frac{\pi}{4} \quad [= 45^\circ]$	$\frac{\pi}{3} \quad [= 60^\circ]$	$\frac{\pi}{2} \quad [= 90^\circ]$
$\sin(\alpha)$	$0 = \frac{\sqrt{0}}{2}$	$\frac{1}{2} = \frac{\sqrt{1}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$1 = \frac{\sqrt{4}}{2}$
$\cos(\alpha)$	$1 = \frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2} = \frac{\sqrt{1}}{2}$	$0 = \frac{\sqrt{0}}{2}$

One can read from the table a mnemonic for the sine and cosine values of the angles at the given places, i.e. 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$. For the sine values, one can insert $k = 0$ to $k = 4$ at $\frac{\sqrt{k}}{2}$ successively; for the cosine, it is exactly the opposite, i.e. starting from $k = 4$ and then descending to $k = 0$. Further values can be obtained by mirroring on the *x-* or *y-axis*.

Theorem 3.1
(Addition theorems) For any angle φ, θ, the following holds:

1.
$$\sin(\varphi \pm \theta) = \sin(\varphi)\cos(\theta) \pm \sin(\theta)\cos(\varphi) \tag{3.6}$$

2.
$$\cos(\varphi \pm \theta) = \cos(\varphi)\cos(\theta) \mp \sin(\theta)\sin(\varphi) \tag{3.7}$$

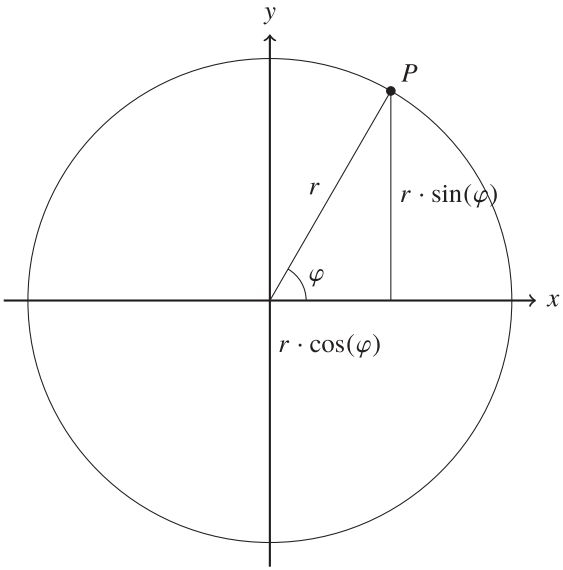
If we set $\varphi = \theta$, we can get two more statements:

$$\sin(2\varphi) = 2\sin(\varphi)\cos(\varphi) \tag{3.8}$$

$$\cos(2\varphi) = \cos(\varphi)^2 - \sin(\varphi)^2 \tag{3.9}$$

3.2 The Polar Form

We know from Sect. 2.3 that the complex number $z = a + \mathrm{i}b$ can be represented as a vector from the origin with endpoint P at the Cartesian coordinates (a, b).



To describe this point, we introduce polar coordinates (r, φ), see figure. Here r is the distance of the point $P = a + \mathrm{i}b$ to the origin and φ is the angle to the positive *x-axis*. Using a rectangular triangle, this gives a direct relationship between (a, b) and (r, φ):

$$a = r\cos(\varphi), \quad b = r\sin(\varphi) \tag{3.10}$$

or vice versa (according to the Pythagorean theorem or trigonometry, see Sect. 3.1):

$$r = \sqrt{a^2 + b^2}, \quad \tan(\varphi) = \frac{b}{a} \quad \Leftrightarrow \varphi = \arctan\left(\frac{b}{a}\right) \tag{3.11}$$

With this, the polar form can now be introduced:

$$z = a + \mathrm{i}b = r\cos(\varphi) + \mathrm{i}r\sin(\varphi) = r(\cos(\varphi) + \mathrm{i}\sin(\varphi)) \tag{3.12}$$

Usually, the angle φ is given in radians and lies in the interval $0 < \varphi < 2\pi$. Due to the 2π-periodicity of sine and cosine, it holds that $\varphi + 2k\pi$, $k \in \mathbb{Z}$, is equivalent to φ.

Polar form

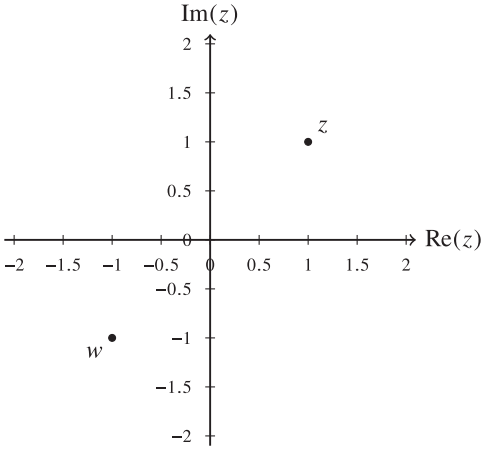
For $z = a + \mathrm{i}b$, applies

$$z = r(\cos(\varphi) + \mathrm{i}\sin(\varphi)) \tag{3.13}$$

with $r = |z| = \sqrt{a^2 + b^2}$ and $\varphi = \arg(z) = \arctan\left(\frac{b}{a}\right)$. In this case, r is called the **magnitude** and φ the **angle** or **argument of** the complex number z. This representation is called the **polar form**.

Represent $z = 1 + \mathrm{i}$ and $w = -1 - \mathrm{i}$ in polar form.

For z and w, it holds that $r = \sqrt{1^2 + 1^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$. Also, we can calculate that in both cases $\tan(\varphi) = \frac{1}{1} = \frac{-1}{-1} = 1$. Now, we must consider that the tangent, being the quotient of sine and cosine, is a π-periodic function. We must, therefore, consider in which quadrants z and w lie:



If $\tan(\varphi) = 1$, it means that $\sin(\varphi) = \cos(\varphi)$. This is exactly true for the two angles $\varphi_1 = \frac{\pi}{4}$ and $\frac{5\pi}{4}$. Using the notion in the coordinate system, we see that the angle is $\frac{\pi}{4}$ at z and it is $\frac{5\pi}{4}$ at w. Thus, holds: $z = \sqrt{2}\big(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\big)$ and $w = \sqrt{2}\big(\cos\left(\frac{5\pi}{4}\right) + \mathrm{i}\sin\left(\frac{5\pi}{4}\right)\big)$.

Typical error

Under the root, there is always a sum of positive numbers: For $w = -1 - \mathrm{i}$, the expression $\sqrt{(-1)^2 + (-\mathrm{i})^2} = 0$ is sometimes calculated incorrectly instead of $r = \sqrt{(-1)^2 + (-1)^2}$, that is, i is incorrectly drawn into the imaginary part. As explained in Sect. 2.1, this is a real number, namely the number that is multiplied by i in the Cartesian representation.

3.2.1 Multiplication and Division in Polar Form

So far, the polar form does not show any advantages. In particular, the calculation of $\arg(z)$ can be difficult. Nevertheless, the polar form of a complex number is often used for the multiplication and division of complex numbers. For this purpose, we consider two complex numbers in polar form:

Theorem 3.2
Let $z = r(\cos(\varphi) + \mathrm{i}\sin(\varphi))$, $w = s(\cos(\theta) + \mathrm{i}\sin(\theta)$ be two complex numbers. Then, we get:

(a)

$$z \cdot w = rs(\cos(\varphi + \theta) + \mathrm{i}\sin(\varphi + \theta)) \tag{3.14}$$

(b)

$$\frac{z}{w} = \frac{r}{s}(\cos(\varphi - \theta) + \mathrm{i}\sin(\varphi - \theta)) \tag{3.15}$$

Proof
For (a), the product is the following, using the addition theorems (3.6) and (3.7) in the last step:

$$\begin{aligned} z \cdot w &= (r(\cos(\varphi) + \mathrm{i}\sin(\varphi)) \cdot (s(\cos(\theta) + \mathrm{i}\sin(\theta))) \\ &= rs(\cos(\varphi)\cos(\theta) - \sin(\varphi)\sin(\theta) + \mathrm{i}(\sin(\varphi)\cos(\theta) + \cos(\varphi)\sin(\theta))) \\ &= rs(\cos(\varphi + \theta) + \mathrm{i}\sin(\varphi + \theta)) \end{aligned}$$

For (b), the proof works analogously. □.

> Multiplication (or division) of complex numbers in polar form

In the product of two complex numbers in polar form, the absolute values r and s are multiplied and the angles φ and θ are added. In division, the absolute values r and s are divided and the angles φ and θ are subtracted.

? Let $z = 2\left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right)$ and $w = 3\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$. What is $z \cdot w$?

$$\begin{aligned} z \cdot w =& 2\left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right) \cdot 3\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right) \\ =& 2 \cdot 3 \cdot \left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right) \cdot \left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right) \\ =& 6 \cdot \left(\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)\right) \\ =& 6 \cdot \left(\cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)\right) = 6\mathrm{i} \end{aligned}$$

3.2.2 Exponentiation of Complex Numbers in Polar Form

For the sake of simplicity, let us first consider a complex number in polar form with magnitude 1, i.e., $z = \cos(\varphi) + \mathrm{i}\sin(\varphi)$. When exponentiating, the magnitude is exponentiated, see Sect. 3.2.1. When considering the angle φ, we will see that exponentiation here causes movements on the unit circle.

Theorem 3.3
It applies:

$$(\cos(\varphi) + \mathrm{i}\sin(\varphi))^n = \cos(n\varphi) + \mathrm{i}\sin(n\varphi) \tag{3.16}$$

Proof
We first examine the statement for $n = 2$:

$$\begin{aligned} (\cos(\varphi) + \mathrm{i}\sin(\varphi))^2 &= (\cos(\varphi) + \mathrm{i}\sin(\varphi)) \cdot (\cos(\varphi) + \mathrm{i}\sin(\varphi)) \\ &= \cos(\varphi)^2 + \mathrm{i}\cos(\varphi)\sin(\varphi) + \mathrm{i}\cos(\varphi)\sin(\varphi) + \mathrm{i}^2\sin(\varphi)^2 \\ &= \cos(\varphi)^2 - \sin(\varphi)^2 + 2\mathrm{i}\cos(\varphi)\sin(\varphi) \\ &\overset{(3.8),(3.9)}{=} \cos(\varphi + \varphi) + \mathrm{i}\sin(\varphi + \varphi) = \cos(2\varphi) + \mathrm{i}\sin(2\varphi) \end{aligned}$$

For $n = 3$, we get:

$$\begin{aligned}(\cos(\varphi) + \mathrm{i}\sin(\varphi))^3 &= (\cos(\varphi) + \mathrm{i}\sin(\varphi))^2 \cdot (\cos(\varphi) + \mathrm{i}\sin(\varphi)) \\ &= (\cos(2\varphi) + \mathrm{i}\sin(2\varphi)) \cdot (\cos(\varphi) + \mathrm{i}\sin(\varphi)) \\ &= \cos(2\varphi)\cos(\varphi) + \mathrm{i}\cos(2\varphi)\sin(\varphi) + \mathrm{i}\cos(\varphi)\sin(2\varphi) + \mathrm{i}^2\sin(2\varphi)\sin(\varphi) \\ &= \cos(2\varphi)\cos(\varphi) - \sin(2\varphi)\sin(\varphi) + \mathrm{i}(\cos(2\varphi)\sin(\varphi) + \cos(\varphi)\sin(2\varphi)) \\ &\overset{(3.6),(3.7)}{=} \cos(2\varphi + \varphi) + \mathrm{i}\sin(2\varphi + \varphi) = \cos(3\varphi) + \mathrm{i}\sin(3\varphi)\end{aligned}$$

Analogously, one can show this statement now for any $n \in \mathbb{N}$. □

? Let $z = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}$. What are z^2 and z^4 and where do these two numbers lie in the Gaussian number plane?

- $$z^2 = \left(\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + 2 \cdot \frac{1}{2}\mathrm{i} - \frac{1}{2} = \mathrm{i} \tag{3.17}$$

  Now, if one draws the numbers on the unit circle, z is on the bisector in the first quadrant and i directly on the positive *y-axis*. Looking at the angle φ, you can see that it doubles as you go from z to z^2.

- $$z^4 = z^2 \cdot z^2 = \mathrm{i}^2 = -1: \tag{3.18}$$

  Plotting shows that we are now on the negative *x-axis* and thus the angle has doubled again.

By the way, we just got to know a square root of i with z. We will deepen these considerations in Chapter 4.

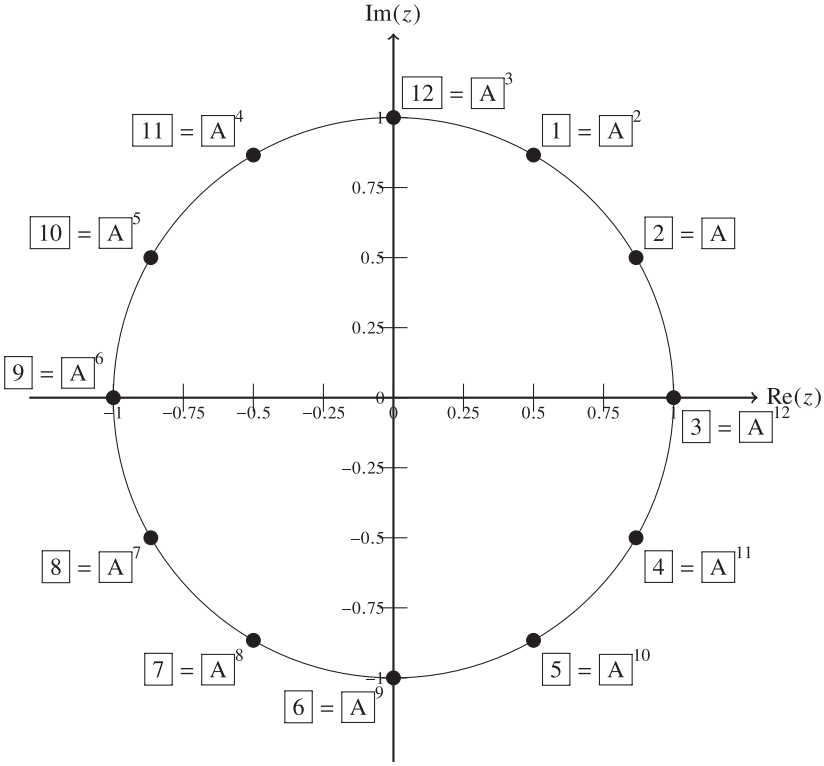
? What are the powers of $\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}$?

By Theorem 3.3, when exponentiating on the left, all multiples of $\frac{\pi}{6}$ are passed through as angles.

In the following, we take the picture of an analogue clock as an aid: If the Gaussian number plane is placed on an analogue clock so that the *x-axis* connects the hour numbers $\boxed{3}$ and $\boxed{9}$ and the *y-axis* connects the numbers $\boxed{6}$ and $\boxed{12}$,

then the number $\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)$ given in the task lies exactly at the $\boxed{2}$ with a distance of 1 from the point of intersection of the *x*- and *y-axes,* i.e. the origin of the coordinate system. (According to Sect. 3.1, $\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\end{pmatrix}$ provides exactly the point on the unit circle, which has the angle φ to the *x-axis*).

When exponentiating, the hour numbers are now passed through backwards, i.e., $\boxed{2}\,\boxed{1}\,\boxed{12}\,\boxed{11}\,\boxed{10}$ etc. Due to the 2π-periodicity of sine and cosine, exponentiation causes the hour hands of the clock to run counterclockwise over and over again. In the following we put for clarity the indicated place, which lies with $\boxed{2}$, as $\boxed{\mathrm{A}}$, thus: $\boxed{\mathrm{A}} := \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$.



We now verify the described motion on the clock or unit circle for the Cartesian representation at the first four powers:

1. According to the task (and you can check it using the angle table from Sect. 3.1), the following applies:

$$\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i} = \cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right) \tag{3.19}$$

2. $\boxed{\mathrm{A}}^2$:

$$\begin{aligned}\boxed{\mathrm{A}}^2 = \boxed{\mathrm{A}} \cdot \boxed{\mathrm{A}} &= \frac{3}{4} - \frac{1}{4} + 2 \cdot \frac{\sqrt{3}}{4}\mathrm{i} = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i} \\ &= \cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\end{aligned} \tag{3.20}$$

3. $\boxed{\mathrm{A}}^3$:

$$\begin{aligned}\boxed{\mathrm{A}}^3 = \boxed{\mathrm{A}}^2 \cdot \boxed{\mathrm{A}} &= \frac{\sqrt{3}}{4} + \frac{1}{4}\mathrm{i} + \frac{3}{4}\mathrm{i} - \frac{\sqrt{3}}{4} = \mathrm{i} \\ &= \cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)\end{aligned} \tag{3.21}$$

4. $\boxed{\mathrm{A}}^4$:

$$\begin{aligned}\boxed{\mathrm{A}}^4 = \boxed{\mathrm{A}}^3 \cdot \boxed{\mathrm{A}} &= \mathrm{i} \cdot \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i} \\ &= \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)\end{aligned} \tag{3.22}$$

Thus, the described movement is also verified for the first powers of the right side. Hence, exponentiation of $\boxed{\mathrm{A}}$ describes a counterclockwise movement on the hour hands along the unit circle. More about this in Chapter 4!

We also found a complex number $\boxed{\mathrm{A}}$ whose third power is i:

Considering power laws and the already described powers of i, we get

$$\boxed{\mathrm{A}}^6 = \mathrm{i}^2 = -1, \quad \boxed{\mathrm{A}}^9 = \mathrm{i}^3 = -\mathrm{i} \text{ and } \boxed{\mathrm{A}}^{12} = \mathrm{i}^4 = 1 \tag{3.23}$$

This is—similar to the powers of i- extendable to

$$\begin{aligned}\boxed{\mathrm{A}}^{3+12k} &= \mathrm{i}, \quad \boxed{\mathrm{A}}^{6+12k} = \mathrm{i}^2 = -1, \quad \boxed{\mathrm{A}}^{9+12k} = \mathrm{i}^3 = -\mathrm{i} \text{ and } \boxed{\mathrm{A}}^{12+12k} \\ &= \mathrm{i}^4 = 1\end{aligned} \tag{3.24}$$

($k \in \mathbb{Z}$). By multiplying by $\boxed{\mathrm{A}}$ or $\boxed{\mathrm{A}}^2$ we get all further powers of $\boxed{\mathrm{A}}$.

3.3 Relationship Between Exponential Functions and Trigonometric Functions

In the following, the connection between polar form and Euler form is shown. For a deeper understanding, the theory of series is required.

Justification of the representation

Using Taylor series with a development point $x_0 = 0$, the exponential, sine, and cosine functions can be represented as follows:

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{3.25}$$

By insertion we get:

$$\begin{aligned}
\cos(x) + \mathrm{i}\sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + \mathrm{i} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^2}{4!} - \frac{x^6}{2!} \pm \cdots\right) \pm \mathrm{i}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \cdots\right) \\
&= \frac{(ix)^0}{0!} + \frac{(ix)^1}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots \\
&= \mathrm{e}^{ix}
\end{aligned}$$

Definition 3.2
Using the polar form from the previous chapter, we can write shorter:

$$\mathrm{e}^{\mathrm{i}\varphi} = \cos(\varphi) + \mathrm{i}\sin(\varphi) \tag{3.26}$$

The complex exponential function is 2π-periodic

The definition shows that the complex exponential function $\mathrm{e}^{\mathrm{i}\varphi}$ differs significantly from the real exponential function e^x. The real exponential function is monotonically increasing and has no function value for any exponent that is less than or equal to 0. Neither of these applies to the complex exponential function,

which, unlike the real exponential function, is even periodic due to its relationship to sine and cosine.

? How can $e^{i\pi}$ and $e^{i\cdot\frac{\pi}{3}}$ be converted into the other two representations?

a) $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i \cdot 0 = -1$
b) $e^{i\cdot\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) + i \cdot \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

3.4 The Euler Form

Using $z = r(\cos(\varphi) + i\sin(\varphi))$ and $e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$, we obtain another form to describe a complex number: $z = re^{i\varphi}$, the so-called Euler form.

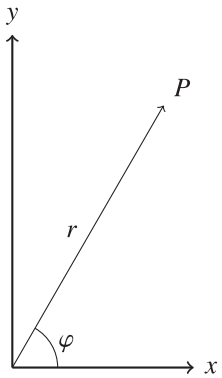
> Euler form

The **Euler form of** a complex number is

$$z = re^{i\varphi} \text{ with } r = |z| \text{ and } \varphi = \arg(z), \tag{3.27}$$

viz.

$$z = re^{i\varphi} = r(\cos(\varphi) + i\sin(\varphi)) \tag{3.28}$$



? Express $z = 4e^{i\frac{\pi}{4}}$ in Cartesian form

$$z = 4 \cdot e^{i\frac{\pi}{4}} = 4 \cdot \left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = 4 \cdot \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 2\sqrt{2} + 2i\sqrt{2} \tag{3.29}$$

We now consider how various definitions already considered in the Cartesian representation and the polar form can be adapted to the Euler form: Let $z = re^{i\varphi}$ and $w = se^{i\theta}$. What is z^{-1}, $\overline{z}$ and $z \cdot w$ then?

a) According to power laws, z^{-1}:

$$z^{-1} = \frac{1}{re^{i\varphi}} = \frac{1}{r}e^{-i\varphi} \tag{3.30}$$

b) For the complex conjugate $\overline{z}$, we obtain using the polar form $z = re^{i\varphi} = r(\cos(\varphi) + i\sin(\varphi))$. Then, in polar form:

$$\overline{z} = r(\cos(\varphi) - i\sin(\varphi)) \tag{3.31}$$

Now, we can use the symmetry properties of sine and cosine, so $\sin(-x) = -\sin(x)$ and $\cos(x) = \cos(-x)$. This gives us:

$$\overline{z} = r(\cos(\varphi) - i\sin(\varphi)) = r(\cos(-\varphi) + i\sin(-\varphi)) = re^{-i\varphi} \tag{3.32}$$

It turns out: we can replace the i by $-$i analogous to the Cartesian form.

c)
$$z \cdot w = \left(re^{i\varphi}\right) \cdot \left(se^{i\theta}\right) = rs \cdot e^{i\varphi+i\theta} = rs \cdot e^{i(\varphi+\theta)} \tag{3.33}$$

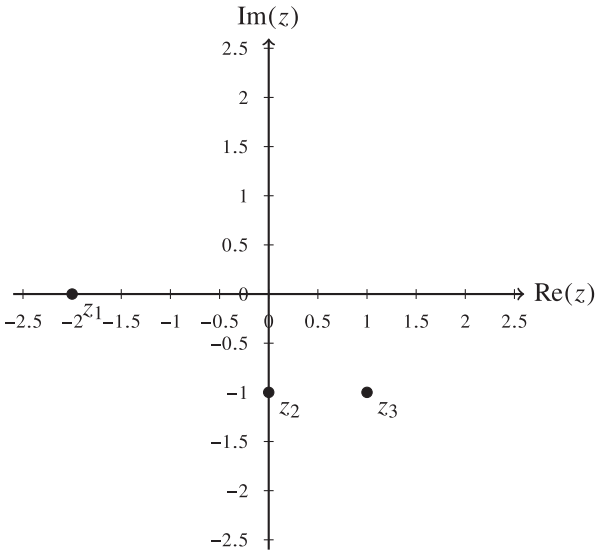
This again applies according to power laws.

We can also do the math:

$$z \cdot \overline{z} = re^{i\varphi} \cdot re^{-i\varphi} = r^2e^{i\varphi-i\varphi} = r^2e^0 = r^2 \tag{3.34}$$

This is correct since in Cartesian form $z \cdot \overline{z} = a^2 + b^2$ is valid and again $r = \sqrt{a^2 + b^2}$ is according to the definition of the Euler form.

? How can the following complex numbers in Cartesian form be converted into Euler form? $z_1 = -2$, $z_2 = -i$ and $z_3 = 1 - i$



- The number z_1 has a distance of 2 to the origin. Since the distance is always non-negative, we must get the minus sign using the angle. The angle to the negative *x-axis* on which -2 lies is π. This gives us a total of: $z_1 = 2\mathrm{e}^{\mathrm{i}\pi}$.
- The number z_2 has a distance of 1 from the origin. Analogous to z_1, the distance is not negative. Since we are on the negative *y-axis*, we have an angle of $\frac{3}{2}\pi$ from the positive *x-axis*, viewed counterclockwise. This gives us a total of: $z_2 = \mathrm{e}^{\frac{3}{2}\mathrm{i}\pi}$.
- According to the formula for *r,* we get: $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. The number is on the bisector of the fourth quadrant, which means an angle of $\frac{7}{4}\pi$ from the positive x-axis in a counterclockwise direction. This gives: $z_3 = \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{7}{4}\pi}$.

Check for updates

4 Complex Root Extraction—Moivre's Theorem

4.1 Preliminary Considerations

? For which two numbers does $z^2 = 3 + 4\mathrm{i}$ apply?

To solve this problem, we can assume z to be general in Cartesian representation as $z = a + \mathrm{i}b$. From this follows:

$$z^2 = (a + \mathrm{i}b)^2 = a^2 + 2ab\mathrm{i} + \mathrm{i}^2 b = (a^2 - b^2) + 2ab\mathrm{i} \tag{4.1}$$

By comparing the coefficients of the real and imaginary parts we get $a^2 - b^2 = 3$ and $2ab = 4$. Thus, we obtain directly both solutions, namely $a = 2$ and $b = 1$ or $a = -2$ and $b = -1$. Analogously to the solutions of $\sqrt{a^2}$ in the real numbers, we recognize:

$$z^2 = 3 + 4\mathrm{i} \Leftrightarrow z = \pm(2 + \mathrm{i}) \tag{4.2}$$

However, this way is generally very tedious and requires the application of the binomial theorem (Theorem 2.3), for complex numbers, which leads to more and more summands for higher exponents. For this reason, we now deal with the other two representations in connection with roots.

? What are the solutions of $x^{12} - 1 = 0$?

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We determined the following decomposition in Chap. 1:

$$\left(x^2-1\right)\left(x^2+1-x\right)\left(x^2+1+x\right)\left(x^2+1\right)\left(x^2+1-\sqrt{3}x\right)\left(x^2+1+\sqrt{3}x\right) \tag{4.3}$$

So, in the chapter, the two real zeros of x^2-1, i.e., $x_{1,2}=\pm 1$ could be determined, but due to negative numbers among the roots, no further ones could be determined. We can continue with the knowledge from the further chapters: Using the *pq formula,* we can now decompose any quadratic polynomial into linear factors. We can consider all solutions as the number of hours of an analogous clock, which is indicated by the numbers in the boxes analogous to the task from Sect. 3.2:

$$x^2-1: x_{1,2}=\pm 1 \quad \boxed{3}, \boxed{9} \tag{4.4}$$

$$x^2+1: x_{3,4}=\pm \mathrm{i} \quad \boxed{12}, \boxed{6} \tag{4.5}$$

$$x^2-x+1: x_{5,6}=\frac{1}{2}\pm\sqrt{\left(\frac{1}{2}\right)^2-1}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i} \quad \boxed{1}, \boxed{5} \tag{4.6}$$

$$x^2+x+1: x_{7,8}=-\frac{1}{2}\pm\sqrt{\left(-\frac{1}{2}\right)^2-1}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i} \quad \boxed{11}, \boxed{7} \tag{4.7}$$

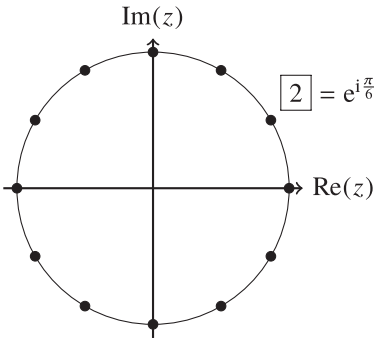
$$x^2-\sqrt{3}x+1: x_{9,10}=\frac{\sqrt{3}}{2}\pm\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2-1}=\frac{\sqrt{3}}{2}\pm\frac{1}{2}\mathrm{i} \quad \boxed{2}, \boxed{4} \tag{4.8}$$

$$x^2+\sqrt{3}x+1: x_{11,12}=-\frac{\sqrt{3}}{2}\pm\sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2-1}=-\frac{\sqrt{3}}{2}\pm\frac{1}{2}\mathrm{i} \quad \boxed{10}, \boxed{8} \tag{4.9}$$

Using the equation $\mathrm{e}^{\mathrm{i}x}=\cos(x)+\mathrm{i}\sin(x)$, one can now convert the twelve values into Euler form. Example:

$$\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\Rightarrow\cos(x)=\frac{\sqrt{3}}{2},\quad \sin(x)=\frac{1}{2}\mathrm{i}\Rightarrow x=\frac{\pi}{6} \tag{4.10}$$

So, we get: $\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}=\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}$ at $\boxed{2}$, see the task from Chap. 3.



The others in the usual representation starting from the positive *x-axis:*

$\boxed{2}\ e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$	$\boxed{1}\ e^{i\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$	$\boxed{12}\ e^{i\frac{\pi}{2}} = i$	$\boxed{11}\ e^{i\frac{2\pi}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
$\boxed{10}\ e^{i\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$	$\boxed{9}\ e^{i\pi} = -1$	$\boxed{8}\ e^{i\frac{7\pi}{6}} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$	$\boxed{7}\ e^{i\frac{4\pi}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$
$\boxed{6}\ e^{i\frac{3\pi}{2}} = -i$	$\boxed{5}\ e^{i\frac{5\pi}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$	$\boxed{4}\ e^{i\frac{11\pi}{6}} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$	$\boxed{3}\ e^{i\frac{12\pi}{6}} = e^0 = 1$

4.2 The Moivre Theorem

We have already seen in Chap. 3 that in the polar form the exponentiation of a complex number by n has the effect that the magnitude is exponentiated by n and the angle is n-folded. We can also use this result for *n-th* roots by exponentiating with $\frac{1}{n}$, which has the following effect analogous to the considerations from Chap. 3 in the example $n = \frac{1}{2}$:

$$\sqrt{\cos(\varphi) + i\sin(\varphi)} = (\cos(\varphi) + i\sin(\varphi))^{\frac{1}{2}} = \cos\left(\frac{1}{2}\varphi\right) + i\sin\left(\frac{1}{2}\varphi\right) \quad (4.11)$$

Using $\cos(\varphi) + i\sin(\varphi) = e^{i\varphi}$, this statement corresponds to a power law:

$$\sqrt{e^{i\varphi}} = \left(e^{i\varphi}\right)^{\frac{1}{2}} = e^{i\frac{\varphi}{2}} \quad (4.12)$$

Analogously, of course, this also applies to arbitrary exponents n, i.e.: $\left(e^{i\varphi}\right)^n = e^{in\varphi}$.

We are now concerned with taking roots from complex numbers, that is, we want to investigate what values $\sqrt[n]{a}$ have, where a is a complex number. We can

reformulate this to ask for which z equations of the form $z^n - a = 0$ are satisfied, for example, the equation $z^3 + 216 = 0$. With our preliminary considerations, there must be a total of n different solutions. We know something similar from the real numbers: The equation $z^2 - 9 = 0$ has exactly $n = 2$ solutions with $z = 3$ and $z = -3$. To find these solutions, we need to examine the argument of z more closely, and the approach can also be applied, for example, to the real case mentioned above.

We already know that, according to our definition, $\varphi = \arg(z)$ is the angle between the *x-axis* and the straight line to the two-dimensional point is *P,* see Sect. 2.3. We also know that, because of the 2π-periodicity of sine and cosine, we can increase the angle φ by 2π and thus obtain the same vector in the Gaussian number plane. More generally, any integer multiple 2π can be added or subtracted without changing the Cartesian form of the complex number.

Uniqueness of $\arg(z)$

$\arg(z)$ is unambiguous to multiples of 2π in radians.

From the previous chapters, we know the polar form of $z = -1 + \mathrm{i}$, viz.

$$z = -1 + \mathrm{i} = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right) \tag{4.13}$$

However, this number is also represented by

$$z = -1 + \mathrm{i} = \sqrt{2}\left(\cos\left(\frac{3\pi}{4} + 2\pi\right) + \mathrm{i}\sin\left(\frac{3\pi}{4} + 2\pi\right)\right) \tag{4.14}$$

while the Cartesian form remains identical. Or more generally because of the 2π -periodicity of sine and cosine, see Eq. (3.3):

$$z = -1 + \mathrm{i} = \sqrt{2}\left(\cos\left(\frac{3\pi}{4} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{3\pi}{4} + 2k\pi\right)\right) \text{ for } k \in \mathbb{N} \tag{4.15}$$

With these preliminary considerations, we now obtain Moivre's theorem:

Theorem 4.1
(Moivre's theorem) For $z = |z|(\cos(\varphi) + \mathrm{i}\sin(\varphi))$ and $k \in \mathbb{N}$, holds

$$z^k = |z|^k(\cos(k\varphi) + \mathrm{i}\sin(k\varphi)) = |z|\mathrm{e}^{ik\varphi} \tag{4.16}$$

This gives the n different roots from z:

$$z_k = \sqrt[n]{|z|}\left(\cos\left(\frac{\varphi + 2k\pi}{n}\right) + \mathrm{i}\sin\left(\frac{\varphi + 2k\pi}{n}\right)\right) = \sqrt[n]{|z|}\ \mathrm{e}^{\mathrm{i}\frac{\varphi+2k\pi}{n}}, k = 0, \ldots, n-1 \quad (4.17)$$

? Determine all solutions of $z^3 + 216 = 0$.

After the preliminary considerations in Chap. 1, we look for here altogether three different solutions. By transformation we get $z^3 = -216$ and can now write the right side as a complex number in polar form:

$$z^3 = 216(\cos(\pi) + \mathrm{i}\sin(\pi)) \text{ , also } r = \sqrt{(-216)^2 + 0^2} = 216,\ \arg(-216) = \pi, \quad (4.18)$$

since -216 lies on the negative *x-axis.*

We can generalize this expression by adding integer multiples of 2π in the argument. This gives us:

$$z^3 = 216(\cos(\pi + 2k\pi) + \mathrm{i}\sin(\pi + 2k\pi)), k \in \mathbb{Z} \quad (4.19)$$

By taking the third root, we now obtain the three solutions using Moivre's theorem:

$$z_{0,1,2} = \sqrt[3]{216}(\cos(\pi + 2k\pi)) + \mathrm{i}\sin(\pi + 2k\pi))^{1/3} \quad (4.20)$$

$$= \sqrt[3]{216}\left(\cos\left(\frac{\pi + 2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi + 2k\pi}{3}\right)\right) \quad (4.21)$$

Now, we can use three successive values of k according to Moivre's theorem. Usually, we use $k = 0$ here and afterwards a suitable number of values in ascending order, so here still $k = 1$ and $k = 2$, because we need three solutions due to the degree of the polynomial and because of the 2π-periodicity of sine and cosine the solutions, for example, $k = 0$ and $k = 3$ or also for $k = 91$ and $k = 142$ are identical. Thus, we obtain:

$$\begin{aligned}
k = 0: &\quad z_0 = 6\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right) = 3\left(1 + \mathrm{i}\sqrt{3}\right)\\
k = 1: &\quad z_1 = 6(\cos(\pi) + \mathrm{i}\sin(\pi)) = -6\\
k = 2: &\quad z_2 = 6\left(\cos\left(\frac{5\pi}{3}\right) + \mathrm{i}\sin\left(\frac{5\pi}{3}\right)\right) = 3\left(1 - \mathrm{i}\sqrt{3}\right)
\end{aligned}$$

It can be calculated by exponentiating by 3 that

$$\left(3\left(1+\mathrm{i}\sqrt{3}\right)\right)^3 = -216 = \left(3\left(1-\mathrm{i}\sqrt{3}\right)\right)^3 \tag{4.22}$$

is. Furthermore, one recognizes that the two complex solutions of the equation are complex conjugate to each other according to the statement from Chap. 2, because $z^3 + 216 = 0$ has only real coefficients.

Instead of using the polar form, we can use the same reasoning for the Euler form. We perform it analogously for the Euler form in the following:

$$\begin{aligned} z^3 = &-216 = 216\mathrm{e}^{\mathrm{i}\cdot\pi} \quad (\text{so}: r = \sqrt{(-216)^2 + 0^2} = 216,\ \arg(-216) = \pi) \\ &= 216\mathrm{e}^{\mathrm{i}\cdot(\pi+2k\pi)} \text{ for } k \in \mathbb{Z} \end{aligned}$$

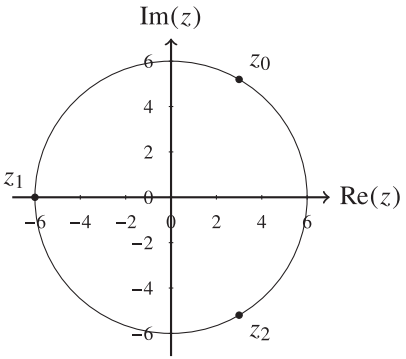
By pulling the third root we get:

$$z = \sqrt[3]{216}\left(\mathrm{e}^{\mathrm{i}\cdot(\pi+2k\pi)}\right)^{\frac{1}{3}} = 6\mathrm{e}^{\frac{\mathrm{i}\cdot(\pi+2k\pi)}{3}} \tag{4.23}$$

Integer values can now be substituted for k, usually using 0 and a suitable number of increasing values (see above):

$$z_0 = 6\mathrm{e}^{\mathrm{i}\cdot\frac{\pi}{3}} = 3\left(1+\mathrm{i}\sqrt{3}\right),\ z_1 = 6\mathrm{e}^{\mathrm{i}\pi} = -6,\ z_2 = 6\mathrm{e}^{\mathrm{i}\cdot\frac{5\pi}{3}} = 3\left(1-\mathrm{i}\sqrt{3}\right) \tag{4.24}$$

These three zeros of $z^3 + 216$ can thus be obtained directly via the Euler form without a detour via the polar form:



Now, it follows another task that works directly with the Euler form and deals more specifically with the transformations of the angles:

? Determine all solutions of $z^5 = 243\mathrm{i}$ in Euler form

For the conversion, $z^5 = 243\mathrm{i} = 243\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$. Thus, we obtain as solutions: $z = \sqrt[5]{243}\mathrm{e}^{\mathrm{i}\varphi_k}$, $k = 0, \ldots, 4$, where holds:

$$5\varphi_0 = \frac{\pi}{2} \Rightarrow \varphi_0 = \frac{\pi}{10},\ 5\varphi_1 = \frac{\pi}{2} + 2\pi \Rightarrow \varphi_1 = \frac{\pi}{2},\ 5\varphi_2 = \frac{\pi}{2} + 4\pi \Rightarrow \varphi_2 = \frac{9\pi}{10},$$

$$5\varphi_3 = \frac{\pi}{2} + 6\pi \Rightarrow \varphi_3 = \frac{13\pi}{10},\ 5\varphi_4 = \frac{\pi}{2} + 8\pi \Rightarrow \varphi_4 = \frac{17\pi}{10}$$

Thus, the searched five solutions of $z^5 = 243\mathrm{i}$ are:

$$z_0 = 3\mathrm{e}^{\mathrm{i}\frac{\pi}{10}},\ z_1 = 3\mathrm{e}^{\mathrm{i}\frac{\pi}{2}},\ z_2 = 3\mathrm{e}^{\mathrm{i}\frac{9\pi}{10}},\ z_3 = 3\mathrm{e}^{\mathrm{i}\frac{13\pi}{10}},\ z_4 = 3\mathrm{e}^{\mathrm{i}\frac{17\pi}{10}} \tag{4.25}$$

What You Learned From This *essential*

- The complex numbers are an extension of the number ranges. Such number range extensions already existed in school: Negative numbers, fractions, real numbers like $\sqrt{2}$ or π.
- There are a total of three representations of complex numbers: Cartesian representation, polar form, and Euler form.
- The different representations are better for different basic arithmetic operations: The Cartesian form is better for addition and subtraction, and the Euler form is better for exponentiation.
- Complex numbers can be represented in all three representations in the Gaussian number plane.
- The topic "complex numbers" has a close relationship to vector calculus, trigonometry and power laws.
- Every *nth* degree polynomial has exactly *n* complex zeros, which can be determined by Moivre's theorem.

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